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On the efficiency of a ducted nonstationary actuator disk

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SUMMARY

In this article an actuator disk with a time-dependent normal load and surrounded by a shroud is discussed. Its efficiency is calculated as a function of its dimensions and frequency. Numerical results are given.

1. Introduction

In 1950 Dickmann [1] wrote a very interesting phenomenological article on several kinds of unsteady propulsion, such as sculling propulsion by wings or by an oscillating piston in a cylinder which induces a periodic jet. In order to obtain insight in the action of these propellers he started with the nonstationary actuator disk and discussed several features of its efficiency. This model was elaborated upon mathematically by Schiele in his doctoral thesis [2] who used the numerical results of his analysis to discuss the efficiency of more general unsteady propellers.

In this paper we give a partial extension of the work of Schiele. The question arises whether the flow induced by an unsteady force action can be smoothed by introducing passive impermeable constraints in the fluid. Then instead of a strongly periodic jet behind the propeller a more constant jet can occur which has the same momentum at the cost of lower kinetic energy losses. Such a device, for instance, is the well-known cylindrical shroud or duct of finite length, used for surrounding ship propellers. By the favourable interaction of the vorticity which leaves the trailing edge of the shroud with the vorticity shed by the propeller the kinetic energy left behind is diminished and the efficiency of the propulsion unit is increased.

In this article the nonstationary actuator disk is surrounded by a shroud which also sheds vorticity from its trailing edge. When the gap between disk and shroud is sufficiently small and the shroud is sufficiently long its effect, as can be expected, is appreciable. The main reason is that a relatively large amount of fluid is "enclosed" by the shroud, which by its inertia smoothes considerably the periodic force action of the disk. Hence even when the forces at the disk are heavily fluctuating the efficiency of disk and shroud together can be high. In fact large forces induced within the shroud at its walls counteract the fluctuating part of the forces of the disk. This means that even in such a case of large fluctuating forces a linear theory is applicable because the disturbance velocities are small.

The efficiency of the actuator disk with time-dependent normal loading surrounded by a shroud is calculated in this article for several values of the width of the gap between disk and shroud and for several values of the length of the shroud. These results show that roughly speaking already for a shroud of a length of a quarter of the length of the disk diameter and for zero gap a remarkable gain in efficiency can be obtained which of course still depends on the frequency and the amplitude of the external force.

2. Formulation of the problem

We consider a cartesian coordinate system (x, y, z) in an unbounded, inviscid and incompressible fluid of density ρ. In the positive x-direction there is a homogeneous flow of magnitude V. The theory is linear; hence the disturbance velocities are assumed to be small of $O(\varepsilon)$ where ε is a small parameter.

The simple propeller model we will use consists of an external force field,

$$\begin{aligned} &\boldsymbol{F}(x, y, z, t) = (F + \Delta F e^{i\omega t})\delta(x)\boldsymbol{e}_x, \quad && y^2 + z^2 \leqslant b_1^2 \\ &\boldsymbol{F}(x, y, z, t) = 0, && y^2 + z^2 > b_1^2 \end{aligned} \tag{2.1}$$

per unit of volume, where F and ΔF are constants of $O(\varepsilon)$, $\boldsymbol{e}_x$ is the unit vector in the x-direction, $\delta(x)$ the delta function of Dirac and b_1 a positive constant. This model is called a nonstationary actuator disk.

We also consider the presence of a rigid and impermeable circle cylinder of finite length l (shroud or duct) which surrounds the disk. It has the same axis as the disk and its radius is b_2 with $b_2 \geqslant b_1$ (Figure 2.1). This shroud extends from $x = x_l$ (leading edge) to $x = x_t$ (trailing edge) with $l = x_t - x_l$.

The linearized equation of motion of the fluid is

$$\frac{\partial \boldsymbol{v}}{\partial t} + V\frac{\partial \boldsymbol{v}}{\partial x} = \frac{1}{\rho}(\boldsymbol{F}(x, y, z, t) - \operatorname{grad} p), \tag{2.2}$$

$\boldsymbol{v}$ being the disturbance velocity and p the pressure. Also we have the equation of continuity

$$\operatorname{div} \boldsymbol{v} = 0. \tag{2.3}$$

First, we consider the free vorticity shed by the actuator disk. Applying the rotation operator to both sides of (2.2) we get

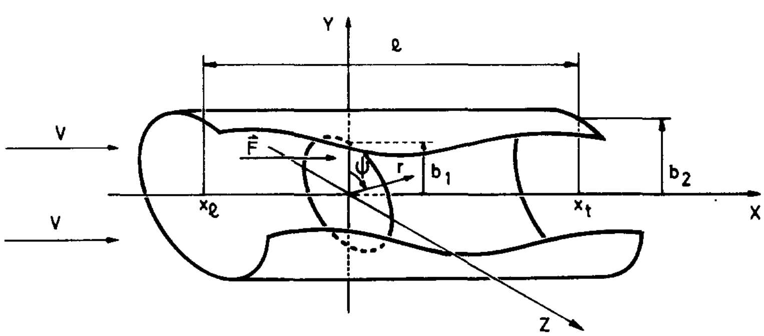

Figure 2.1. Actuator disk and shroud (partially removed) placed in a parallel flow, $l = x_t - x_l$.

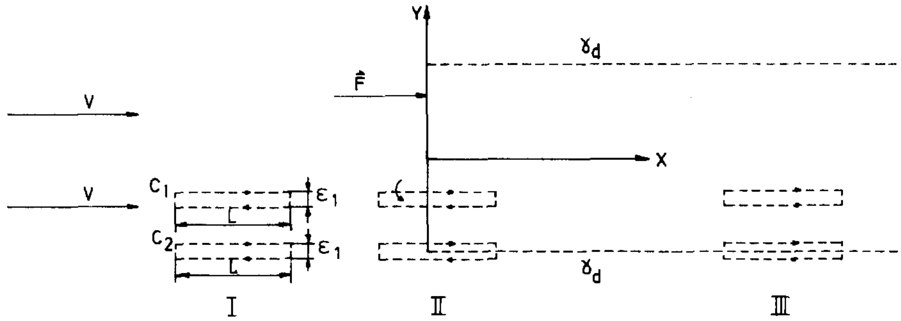

Figure 2.2. Cross section of the disk; testcontours C_1 and C_2floating with the main stream.

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\operatorname{rot} \boldsymbol{v} = \frac{1}{\rho}\operatorname{rot} \boldsymbol{F}. \tag{2.4}$$

It is obvious from the axial symmetry of the problem that the free vorticity will have a component in the Ψ-direction only (Figure 2.1). That's why we use testcontours C_1 and C_2 in the x, y-plane, floating downstream with the velocity V and going to pass the disk (Figure 2.2). The contours have small length L and width ε_1, $\varepsilon_1 \ll L$. In position I the circulations of C_1 and C_2 are zero. In position II it follows from (2.4) and (2.1) for the circulation $\Gamma_{C_1}(t)$ along contour C_1 that

$$\frac{d}{dt}\Gamma_{C_1}(t) = \frac{d}{dt}\int_{C_1} \boldsymbol{v}\cdot\boldsymbol{e}_x dx = \frac{1}{\rho}\int_{C_1} \boldsymbol{F}\cdot\boldsymbol{e}_x dx = 0, \tag{2.5}$$

and analogously for the circulation $\Gamma_{C_2}(t)$ along C_2,

$$\frac{d}{dt}\Gamma_{C_2}(t) = \frac{1}{\rho}(F + \Delta F\,\mathrm{e}^{i\omega t}). \tag{2.6}$$

At the instant the contours C_1 and C_2 are just downstream from the disk their circulations have become, at least if L is sufficiently small,

$$\Gamma_{C_1}(t) = 0,\ \Gamma_{C_2}(t) = \frac{1}{\rho}(F + \Delta F\,\mathrm{e}^{i\omega t})\frac{L}{V}. \tag{2.7}$$

Once the contours have passed the disk there is no change in circulation any more (position III, Figure 2.2). This means that per unit of length in the x-direction the strength of the free vorticity left behind becomes

$$\gamma_d = \frac{\Gamma_{C_2}}{L} = \frac{(F + \Delta F\,\mathrm{e}^{i\omega t})}{\rho V} \tag{2.8}$$

where γ_d is reckoned positive when it is connected with a right-hand screw to the orientation of contours of type C_2. The same definition of positiveness will be assumed in Section 4 for the circular vorticity on the shroud and behind it.

This configuration of ring vortices on the semi-infinite cylinder surface $r = b_1$, $x > 0$ downstream of the edge of the disk (Figure 2.2) will describe the velocity field induced by the external force field $\boldsymbol{F}$. It induces a disturbance velocity with non-zero components normal to the shroud (Figure 2.1). We assume a vortex distribution on the shroud to compensate this normal velocity. As a consequence of the time dependence of the disturbance velocity this vortex distribution will be a function of time too; hence free vorticity will be shed from the trailing edge of the shroud. The interaction of free vorticity behind shroud and disk will influence the efficiency of the system.

In the next section we first calculate the kinetic energy per unit of length in the x-direction of the fluid far downstream when the shroud is absent.

3. The efficiency of the nonstationary actuator disk without shroud ($l = 0$)

It is seen from (2.8) that the free circular vorticity γ_d downstream from the disk at $r = b_1$ (Figure 2.2) has the density

$$\gamma_d(x, t) = \frac{1}{\rho V}(F + \Delta F \mathrm{e}^{i(\omega t - \mu_1 x)}), \qquad \mu_1 = \frac{\omega}{V}. \tag{3.1}$$

We consider the value of the kinetic energy lost per unit of length in the x-direction far behind the disk. Speaking in future about this kinetic energy we think of a disk moving with a velocity V in the negative x-direction in a fluid which is at rest upstream at infinity. Then the shed free vorticity is independent of time and we put $t = 0$. We use the same notation x for the coordinate along the axis of axial symmetry, which will not cause misunderstanding. When we drop one argument in a function we mean that we have removed the factor $\mathrm{e}^{i\omega t}$.

Defining in (3.1) γ_s (steady vorticity) and γ_1 (unsteady vorticity at $r = b_1$) for $x \geqslant 0$ by

$$\gamma_s(x) = \frac{F}{\rho V}, \qquad \gamma_1(x, t) = \frac{\Delta F}{\rho V} \mathrm{e}^{i(\omega t - \mu_1 x)}, \tag{3.2}$$

we can write at $t = 0$

$$\gamma_d(x) = \gamma_s(x) + \gamma_1(x). \tag{3.3}$$

Hence γ_d is a periodic function of x with period $2\pi/\mu_1$. We first calculate the kinetic energy in a region G, defined by

$$A \leqslant x \leqslant B, \qquad 0 \leqslant r, \tag{3.4}$$

for A sufficiently large and $B = A + 2\pi/\mu_1$. By the axial symmetry the velocity field of the disk alone, $\boldsymbol{v}_d$, is a function of x and r only. We split the velocity field $\boldsymbol{v}_d$ into two parts,

$$\boldsymbol{v}_d(x, r) = \boldsymbol{v}_s(x, r) + \boldsymbol{v}_1(x, r), \tag{3.5}$$

where $\boldsymbol{v}_s$ and $\boldsymbol{v}_1$ are the velocity fields induced by the vorticities γ_s and γ_1 respectively.

The velocity field $\boldsymbol{v}_s$ far behind the disk is well known,

$$\boldsymbol{v}_s(x, r) = \begin{cases} \dfrac{F}{\rho V}\boldsymbol{e}_x, & r \leqslant b_1, \\ 0, & r > b_1. \end{cases} \tag{3.6}$$

For x sufficiently large the velocity field $\boldsymbol{v}_1$ is a periodic function of x,

$$\boldsymbol{v}_1(x + 2\pi/\mu_1, r) = \boldsymbol{v}_1(x, r). \tag{3.7}$$

The kinetic energy E in G, belonging to the total velocity field $\boldsymbol{v} = \boldsymbol{v}_s + \boldsymbol{v}_1$, has the value

$$E = \tfrac{1}{2}\rho \iiint_G |\boldsymbol{v}|^2 d\tau. \tag{3.8}$$

Integrating over a length $2\pi/\mu_1$ in the x-direction it can be seen from (3.7) that the contribution of product terms of $\boldsymbol{v}_s$ and $\boldsymbol{v}_1$ in $|\boldsymbol{v}|^2$ vanishes. This enables us to calculate in G the kinetic energies E_s and E_1 of the fields $\boldsymbol{v}_s$ and $\boldsymbol{v}_1$ separately. Simply adding them we get

$$E = E_s + E_1. \tag{3.9}$$

By (3.6) we see that the energy E_s in G is

$$E_s = \tfrac{1}{2}\rho\pi b_1^2 \frac{2\pi}{\mu_1}\left(\frac{F}{\rho V}\right)^2 = \pi^2 \frac{b_1^2}{\mu_1}\frac{F^2}{\rho V^2}. \tag{3.10}$$

The energy E_1 belonging to $\boldsymbol{v}_1$ can be written as

$$E_1 = \tfrac{1}{2}\rho \iint_{S_1} Re(\varphi_1^- - \varphi_1^+)\, Re\left(\frac{\partial \varphi_1}{\partial n}\right) d\sigma, \tag{3.11}$$

where Re means real part, $\varphi_1^+(\varphi_1^-)$ is the potential of the complex velocity field for $r \downarrow b_1$ ($r \uparrow b_1$), $\boldsymbol{n}$ is the unit normal on the boundary S_1, pointing in the direction of increasing r and

$$S_1\colon\ A \leqslant x \leqslant B, \quad r = b_1. \tag{3.12}$$

In arriving at (3.11) we used the periodicity of the velocity field. For the jump in the potential across S_1 we can take

$$\varphi_1^+(x) - \varphi_1^-(x) = \int_x^{x_0} \gamma_1(\xi)d\xi = -\frac{i}{\mu_1}\gamma_1(x) + c_1, \tag{3.13}$$

where x_0 and c_1 are constants, independent of x. The normal derivative $\partial\varphi_1/\partial n$ is independent of Ψ (Figure 2.1), hence we put $\Psi = 0$ and by the Biot–Savart theorem we have

$$\frac{\partial \varphi_1}{\partial n}(x, b_1) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi} \gamma_1(\xi)\frac{(\boldsymbol{e}_\theta \times \boldsymbol{R}, \boldsymbol{e}_y)}{R^3} b_1\, d\theta d\xi = i\mu T(\mu)\gamma_1(x), \tag{3.14}$$

where e_θ and e_y are the unit vectors in θ- and y-direction, in Cartesian coordinates,

$$\boldsymbol{R} = (x - \xi, b_1(1 - \cos\theta), -b_1 \sin\theta), \tag{3.15}$$

R is the length of $\boldsymbol{R}$ and $T(\mu)$ the dimensionless number

$$T(\mu) = \frac{-1}{\pi\mu} \int_0^\infty \int_0^\pi \frac{\eta \sin\mu\eta \cos\theta \, d\theta \, dy}{\{\eta^2 + 2(1 + \cos\theta)\}^{\frac{3}{2}}}, \quad \mu = u_1 b_1 = \frac{\omega b_1}{V}. \tag{3.16}$$

Substitution of (3.13) and (3.14) into (3.11) yields

$$E_1 = \pi^2 \frac{b_1^2}{\mu_1} \frac{(\Delta F)^2}{\rho V^2} T(\mu). \tag{3.17}$$

Per unit of length in the x-direction we find for the values of these energies,

$$E_s^1 = \tfrac{1}{2}\pi b_1^2 \frac{F^2}{\rho V^2}, \quad E_1^1 = \tfrac{1}{2}\pi b_1^2 \frac{(\Delta F)^2}{\rho V^2} T(\mu). \tag{3.18}$$

The useful work W delivered by the disk per unit of length in the x-direction amounts to

$$W = \pi b_1^2 F. \tag{3.19}$$

Hence the efficiency η_1 will be

$$\eta_1 = \frac{W}{W + E_s^1 + E_1^1} = \left\{1 + \frac{F}{2\rho V^2}\left[1 + \left(\frac{\Delta F}{F}\right)^2 T(\mu)\right]\right\}^{-1}. \tag{3.20}$$

Because the theory is linearized $F = O(\varepsilon)$; hence

$$\eta_1 = 1 - d_1\varepsilon, \quad d_1 = \frac{E_s^1 + E_1^1}{W} = \frac{F}{2\rho V^2}\left[1 + \left(\frac{\Delta F}{F}\right)^2 T(\mu)\right]. \tag{3.21}$$

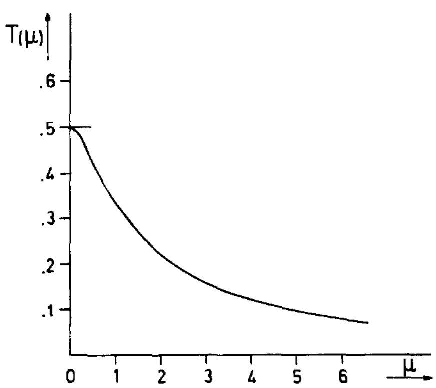

Figure 3.1. $T(\mu)$ as a function of $\mu = \omega b_1/V$.

In Figure 3.1 we have drawn the graph of the dimensionless function $T(\mu)$ which is one of the quantities which determine the efficiency η_1, (3.20) or (3.21).

When $\mu \to 0$, $\gamma_1(x)$ remains periodic, although its period $2\pi/\mu_1$ becomes very large. With $F = \Delta F$, the quantity $T(\mu)$ expresses the ratio of E_1^1 and E_s^1. Therefore we find $T(\mu) = E_1^1/E_s^1 \to 0.5$ when $\mu \to 0$. When $\mu \to \infty$ the period of $\gamma_1(x)$ tends to zero. In those points of G that are at a distance from the cylinder $r = b_1$ which is large with respect to the length of such a small period, the contributions to the velocity field due to positive and negative parts of $\gamma_1(x)$ almost cancel. This explains why $T(\mu) \to 0$ when $\mu \to \infty$. One can easily show that $dT/d\mu \to 0$ when $\mu \to 0$, which also is confirmed by Figure 3.1. It follows from (3.20) or (3.21) that when $\Delta F/F$ and $F/(2\rho V^2)$ are kept constant the efficiency increases with increasing values of $\mu = \omega b_1/V$.

In the next section we shall see how the presence of the shroud (Figure 2.1) affects the efficiency of the system.

4. The actuator disk surrounded by the shroud ($l \neq 0$)

We now assume a shroud of length $l \neq 0$ to be present (Figure 2.1). On and behind it we have a vortex distribution to compensate for the normal components of the velocity induced by the free vorticity $\gamma_d(x, t)$, (3.1). Being interested in the influence of the shroud on the efficiency of the system we only need (because our theory is linear) to consider the vorticity Γ on it due to the time-dependent part of the vorticity $\gamma_d(x, t)$. Per unit of length in the x-direction ($r = b_2$) Γ is written as

$$\Gamma(x, t) = \Gamma(x)e^{i\omega t}, \qquad x_l \leqslant x \leqslant x_t \tag{4.1}$$

with $\Gamma(x) = \Gamma^{(1)}(x) + i\Gamma^{(2)}(x)$, $\Gamma^{(1)}$ and $\Gamma^{(2)}$ real. Consequently at the trailing edge ($x = x_t$) of the shroud free circular vorticity $\gamma_2(x, t)$ will be shed off with density

$$\gamma_2(x_t, t) = -\frac{1}{V}\frac{\partial}{\partial t}\int_{x_l}^{x_t} \Gamma(\xi, t)d\xi = -\frac{i\omega}{V}e^{i\omega t}\int_{x_l}^{x_t} \Gamma(\xi)d\xi \tag{4.2}$$

per unit of length in the x-direction. First we consider the case of a gap $b_2 - b_1 > 0$. By the Kutta condition we find that this shed vorticity at the trailing edge equals $\Gamma(x_t, t)$,

$$\Gamma(x_t) = -\frac{i\omega}{V}\int_{x_l}^{x_t} \Gamma(\xi)d\xi, \qquad b_2 > b_1. \tag{4.3}$$

Hence

$$\gamma_2(x, t) = \Gamma(x_t)e^{i[\omega t - \mu_1(x - x_t)]}, \qquad x \geqslant x_t,\ b_2 > b_1. \tag{4.4}$$

To find $\gamma_2(x_t)$ we have, compensating for the induced normal velocity at points $x = x^+$, $r = b_2$ on the shroud, to solve together with the equation (4.3) the integral equation for $\Gamma(x)$,

$$\oint_{x_l}^{x_t}\int_0^{2\pi} \Gamma(\xi)\frac{(e_\theta \times R_2, e_y)}{R_2^3}\, b_2\, d\theta\, d\xi +$$

$$+\int_{x_t}^{\infty}\int_0^{2\pi}\gamma_2(\xi)\frac{(e_\theta\times \boldsymbol{R}_2, e_y)}{R_2^3}\, b_2\, d\theta\, d\xi$$

$$+\int_0^{\infty}\int_0^{2\pi}\gamma_1(\xi)\frac{(e_\theta\times \boldsymbol{R}_3, e_y)}{R_3^3}\, b_1\, d\theta\, d\xi = 0, \tag{4.5}$$

where $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ are given in Cartesian coordinates by

$$\begin{aligned}\boldsymbol{R}_2 &= (x^+ - \xi,\ b_2(1-\cos\theta),\ -b_2\sin\theta),\\ \boldsymbol{R}_3 &= (x^+ - \xi,\ b_2 - b_1\cos\theta,\ -b_1\sin\theta),\end{aligned} \tag{4.6}$$

and $R_j = |\boldsymbol{R}_j|$, $j = 2, 3$. The first term on the left hand side of (4.5) expresses the normal velocity induced on the shroud by the vorticity $\Gamma(x, t)$ itself. Therefore we have to take the Cauchy principal value.

To calculate the lost kinetic energy E far downstream we again consider the region G (3.4). Also we can again make the addition

$$E = E_s + E_2,$$

where E_s has the same meaning as before and E_2 is the kinetic energy belonging to the two vortex layers γ_1 and γ_2. Analogous to (3.11) the value of E_2 is

$$E_2 = \tfrac{1}{2}\rho\iint_{S_1} Re(\phi^- - \phi^+)Re\left(\frac{\partial\phi}{\partial n_1}\right)d\sigma + \tfrac{1}{2}\rho\iint_{S_2} Re(\phi^- - \phi^+)Re\left(\frac{\partial\phi}{\partial n_2}\right)d\sigma, \tag{4.8}$$

where $\phi^+(\phi^-)$ is the velocity potential of the periodic part of the velocity field for $r\downarrow b_j$ $(r\uparrow b_j)$, $\boldsymbol{n}_j$ the unit normal on the boundary S_j, pointing in the direction of increasing r and

$$S_j\colon\quad A\leqslant x\leqslant B,\quad r = b_j,\ j = 1, 2. \tag{4.9}$$

For the jumps in the potential across S_j we can take similarly as before

$$\begin{aligned}\phi^+(x) - \phi^-(x)|_{r=b_1} &= \frac{-i}{\mu_1}\gamma_1(x) + c_1,\\ \phi^+(x) - \phi^-(x)|_{r=b_2} &= \frac{-i}{\mu_1}\gamma_2(x) + c_2,\end{aligned} \tag{4.10}$$

where c_1 and c_2 are constants independent of x. The normal derivative $\partial\phi/\partial n_j$ can be shown to be

$$\begin{aligned}\frac{\partial\phi}{\partial n_1} &= i\mu\{I_1(1,\mu)\gamma_1(x) + bI_2(b,\mu)\gamma_2(x)\},\\ \frac{\partial\phi}{\partial n_2} &= i\mu\{I_2(b,\mu)\gamma_1(x) + bI_1(b,\mu)\gamma_2(x)\},\end{aligned} \tag{4.11}$$

with b, I_1 and I_2 dimensionless numbers given by

$$I_1(\tau,\mu) = \frac{-1}{\pi\mu}\int_0^\infty\int_0^\pi \frac{\eta \sin\mu\eta \cos\theta d\theta d\eta}{\{\eta^2 + 2\tau^2(1+\cos\theta)\}^{\frac{3}{2}}}, \tag{4.12}$$

$$I_2(b,\mu) = \frac{-1}{\pi\mu}\int_0^\infty\int_0^\pi \frac{\eta \sin\mu\eta \cos\theta d\theta d\eta}{\{\eta^2 + 1 + b^2 + 2b\cos\theta\}^{\frac{3}{2}}}, \quad b = b_2/b_1. \tag{4.13}$$

From the definitions (3.16), (4.12) and (4.13) we see that $I_1(1,\mu) = I_2(1,\mu) = T(\mu)$. Thus the value of this part of the lost kinetic energy per unit of length in the x-direction is:

$$E_2^1 = \tfrac{1}{2}\pi b_1^2 \frac{(\Delta F)^2}{\rho V^2}[I_1(1,\mu) + \beta^2 b^2 I_1(b,\mu) + 2\beta b \cos(\mu_1 x_t + \alpha) I_2(b,\mu)], \tag{4.14}$$

where α and β are the dimensionless numbers

$$\alpha = \arctan(\Gamma^{(2)}(x_t)/\Gamma^{(1)}(x_t)), \tag{4.15}$$

$$\beta = \text{sign}(\Gamma^{(1)}(x_t))\sqrt{\{\Gamma^{(1)}(x_t)\}^2 + \{\Gamma^{(2)}(x_t)\}^2}/[(\Delta F)/\rho V]. \tag{4.16}$$

Now the efficiency η_2 of the nonstationary actuator disk in presence of the shroud is

$$\eta_2 = \frac{W}{W + E_s^1 + E_2^1}. \tag{4.17}$$

Both E_1^1 (3.18) and E_2^1 are the lost kinetic energies due to the unsteady part of $\boldsymbol{F}(x, y, z, t)$ in absence and presence of the shroud respectively. Let us call them the unsteady energies. Then $(E_1^1 - E_2^1)/E_1^1 = 1 - E_2^1/E_1^1$ is the fraction of E_1^1 regained by the shroud. The ratio E_2^1/E_1^1, the fraction of E_1^1 that remains in the fluid, depends on four dimensionless parameters:

$$E_2^1/E_1^1 = f(\mu, b, l', x_t'), \quad b = b_2/b_1 > 1 \tag{4.18}$$

where l' and x_t' are given by

$$l' = l/b_1, \quad x_t' = x_t/b_1. \tag{4.19}$$

Using (4.18) and (3.18) we can write η_2 (4.17) as

$$\eta_2 = \left\{1 + \frac{F}{2\rho V^2}\left[1 + \left(\frac{\Delta F}{F}\right)^2 T(\mu) f(\mu, b, l', x_t')\right]\right\}^{-1} \tag{4.20}$$

or, because the theory is linear

$$\eta_2 = 1 - d_2\varepsilon, \; d_2 = \frac{F}{2\rho V^2}\left[1 + \left(\frac{\Delta F}{F}\right)^2 T(\mu) f(\mu, b, l', x_t')\right]. \tag{4.21}$$

5. The limit case $b_2 - b_1 = 0$ ($l \neq 0$)

In this section we consider the situation in which there is no gap between shroud and disk, hence when $b_2 - b_1 = 0$. It turns out that in this case a change of the position of the actuator disk within the duct does not affect the velocity field induced by the system. This follows from the arguments given in [3] with respect to the deformation of the area of an actuator disk which has a time-dependent loading perpendicular to its surface. This loading has to be independent of the position on the surface of the disk. Both conditions are satisfied in our model. We replace the actuator disk of which a cross section with the x, y-plane is given in Figure 5.1a, by an actuator surface of which the shape is drawn in Figure 5.1b.

In both cases the induced velocity fields are equal. When now we consider the limit $b_2 \to b_1$, the cylindrical part of the actuator surface in Figure 5.1b, coincides with the duct. The duct annihilates the velocities induced by this cylindrical part; hence only an actuator disk inside the duct remains active and it induces the same velocities as the actuator disk at $x = 0$ in Figure 5.1a. This is confirmed numerically in Figure 6.1 where the case of a small gap $b_2/b_1 = 1.05$ is considered.

This property is useful for numerical reasons. It enables us to put in the case $b_2 - b_1 = 0$ the trailing edge of the shroud at $x = 0$ ($x_t = 0$). So we do not have to deal with the numerical problem to determine the vorticity Γ on the shroud having a jump at the position of the disk. Here Γ represents again the vorticity on the duct caused by the unsteady part of the force field. The Kutta condition at the trailing edge for the unsteady part of the vorticity can be written in this case as

$$\Gamma(0) = \gamma_2(0) + \gamma_1(0), \quad b_2 - b_1 = 0. \tag{5.1}$$

Hence for $\gamma_2(x, t)$ we have

$$\gamma_2(x,t) = (\Gamma(0) - \gamma_1(0))e^{i[\omega t - \mu_1 x]}; \quad x \geqslant 0, \ r = b_2 = b_1. \tag{5.2}$$

Substituting this into (4.5) we get the equation

$$\int_{x_l}^{0} \int_{0}^{2\pi} \Gamma(\xi) \frac{(e_\theta \times R_2, e_y)}{R_2^3} d\theta d\xi + \int_{0}^{\infty} \int_{0}^{2\pi} \Gamma(0) e^{-i\mu_1 \xi} \frac{(e_\theta \times R_2, e_y)}{R_2^3} d\theta d\xi = 0. \tag{5.3}$$

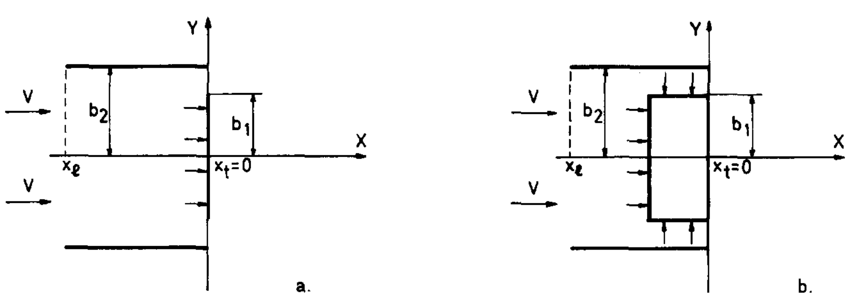

Figure 5.1. Deformation of an actuator disk.

Equation (5.3) has to be solved, together with the equation that follows from (4.2) and (5.1), i.e.

$$\frac{-i\omega}{V}\int_{x_l}^{0}\Gamma(\xi)d\xi = \Gamma(0) - \gamma_1(0). \tag{5.4}$$

The total vorticity behind shroud and disk is

$$\Gamma(0)e^{i[\omega t - \mu_1 x]}, \qquad x \geqslant 0,\ r = b_2 = b_1. \tag{5.5}$$

Analogous to (3.18) we can write for the value of the kinetic energy per unit of length in the x-direction

$$E_2^1 = \tfrac{1}{2}\rho\pi b_1^2\Gamma^2(0)T(\mu), \qquad b = 1. \tag{5.6}$$

The ratio of E_2^1 and E_1^1 (the energies left downstream in presence and in absence of the shroud respectively) now depends on two dimensionless parameters only,

$$E_2^1/E_1^1 = g(\mu, l'), \qquad b = 1,\ x_l \leqslant 0 \leqslant x_t \tag{5.7}$$

The efficiency η_2, in the case $x_l \leqslant 0 \leqslant x_t$, again follows from (4.20) or (4.21) where we have to replace $f(\mu, b, l', x_t')$ by $g(\mu, l') = f(\mu, 1, l', x_t')$.

6. Numerical results

We first recapitulate the meaning of the dimensionless parameters used in Figures 6.1–6.4: $\mu = \omega b_1/V$, $b = b_2/b_1$, $l' = l/b_1$, $x_t' = x_t/b_1$. For some of the parameters with dimension we refer to Figure 2.1.

Figures 6.1–6.4 indicate in which way the quantities f (4.18) and g (5.7) depend on their parameters. As expected we observe from Figure 6.1 that as long as the shroud is far ahead of the disk nearly nothing of the lost kinetic energy E_1^1 is regained (f tends to one). When it is placed far downstream from the disk it tends to its limit value when it encloses the two sided infinite vortex layer (the horizontal dotted lines drawn in Figure 6.1). We see that even for

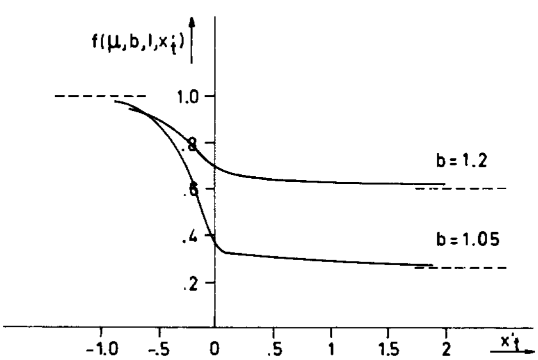

Figure 6.1. f as a function of the position of the trailing edge; $l' = 1$ and $\mu = 2$. The disk is at the origin; —— calculated values, ---- asymptotic values.

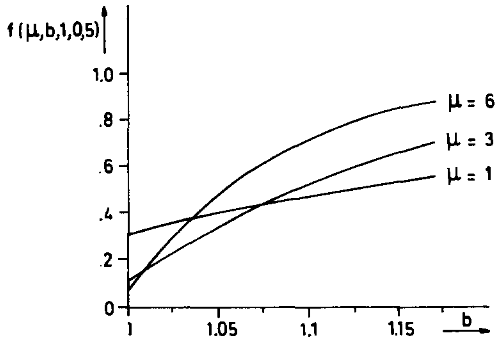

Figure 6.2. f as a function of the gap width.

the given values of b in Figure 6.1 the value f is almost independent of the position of the shroud in case $x_l \leqslant 0 \leqslant x_t$.

Figure 6.2 shows that when we vary b only the fraction f of the unsteady energy E_1^1 that remains increases with increasing values of b. This is a logical consequence of the decreasing interaction of disk and shroud when the gap $b_2 - b_1$ becomes larger.

In Figure 6.3 we see that for a fixed value of b there is a certain value of μ for which relatively most of the lost kinetic energy is regained by the duct; the shroud is working optimal. To this we remark the following: starting at $\mu = 0$ we find that for increasing μ the amplitude $|\Gamma(x_t)|$ of γ_2 (4.4) becomes larger and hence the interaction between the vortex layers is improved. After a certain value of μ (it depends on b) this interaction diminishes because the period of the shed vorticity becomes smaller. Then the distance at which the influence of a sheet can be perceived becomes smaller, hence its favourable interaction with the other sheet becomes less.

A graph of $g(\mu, l')$ is given in Figure 6.4 ($b = 1$, $x_l \leqslant 0 \leqslant x_t$). When $\mu \to 0$ the length of the period of γ_1 tends to infinity. The influence of γ_1 on the shroud for small values of μ tends to the influence of a vortex sheet of constant strength in which case no vorticity is shed by the duct. Hence the duct is not able to regain a part of the energy E_1^1, so $g(\mu, l') \to 1$ for $\mu \to 0$ (l' fixed). The longer the shroud, the better it is able to shed the vorticity γ_2 needed to compensate the vorticity γ_1 behind the edge of the disk. This is numerically confirmed by Figure 6.4 where it is seen that for increasing values of l' the value of g becomes smaller.

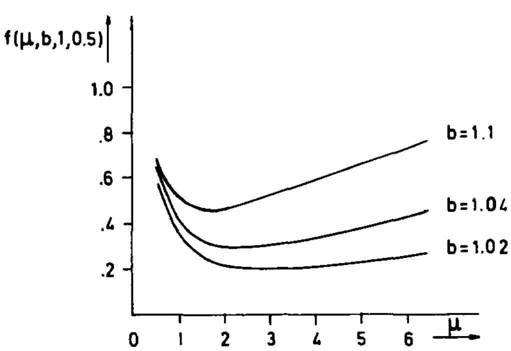

Figure 6.3. f as a function of μ.

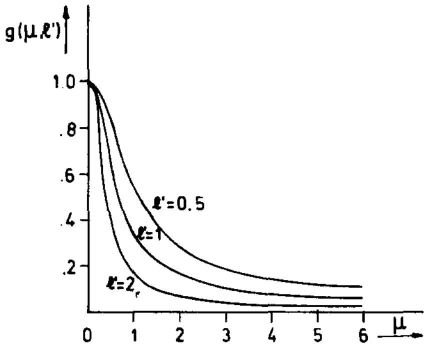

Figure 6.4. g as a function of μ for different values of the length of the shroud.

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